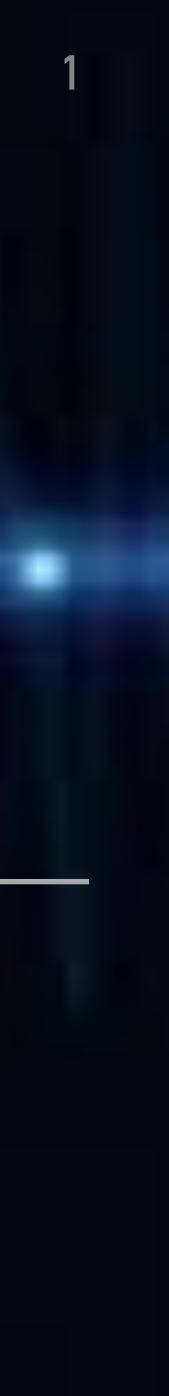
SPARSE RECOVER **SPARSE REPRESENTATIONS**



SPARSE RECOVERY





PROBLEM

- Let $x \in \mathbb{R}^N$ be a *s*-sparse vector, is such that $||x||_0 = s$
- Let $A \in \mathbb{R}^{MN}$ be a measurement matrix, and let

When can x be exactly recovered from y ?

All the presented results and much more can be found in: A Mathematical Introduction to Compressive Sensing, S. Foucart & H. Rauhut

y = Ax



REMINDER: ALGORITHMS

- Matching Pursuit and Orthogonal Matching Pursuit
- Basis Pursuit
- Iterative Thresholding



NULL SPACE PROPERTY

- Let $A \in \mathbb{R}^{MN}$ be a matrix with normalized columns ($||a_i||_2 = 1 \forall i$).
- > A satisfies the null space property (NSP) relative to a set S iff $\|v_{\bar{S}}\|_{1} < \|v_{\bar{S}}\|_{1}$ for all $v \in \ker(A) \setminus \{0\}$
- Figuivalently, $A \in \mathbb{R}^{MN}$ satisfies the NSP relative to a set S iff
 - $2\|v_S\|_1 \le \|v\|_1$ for all $v \in ker(A) \setminus \{0\}$
- A satisfies the null space property of order s if ti satisfies the NSP for all S such that $card(S) \leq s$





NULL SPACE AND S-SPARSE RECOVERY

- y = Ax iff A satisfies the NSP relative to S
- iff A satisfies the NSP of order s

• Given a matrix $A \in \mathbb{R}^{MN}$, every vector $x \in \mathbb{R}^N$ such that supp(x) = S is the unique solution of

• Given a matrix $A \in \mathbb{R}^{MN}$, every vector s—sparse vector $x \in \mathbb{R}^N$ is the unique solution of y = Ax





PROOF

Suppose that for all x supported on S, x is the unique minimizer of $||z||_1 s \cdot t Ax = Az$.

Let $v \in \text{ker}(A) \setminus \{0\}$, v_S is then the unique minimizer of $||z||_1 s \cdot t Av_S = Az$. by unicity of v_S . Then A satisfies the NSP relative to S

Suppose that A satisfies the NSP relative to S. Let x supported on S and a vector $z \neq x$ such that Ax = Az. Then $v = x - z \in \text{Ker}(A) \setminus \{0\}$. Then

 $||x||_1 < ||z_{\bar{S}}||_1 + ||z_S||_1 = ||z||_1$, then x minimizes $||x||_1$

- Moreover, we have $0 = Av = A(v_{\bar{S}} + v_{\bar{S}})$, then $A(-v_{\bar{S}}) = Av_{\bar{S}}$ and necessarily we have $\|v_{\bar{S}}\|_1 < \|v_{\bar{S}}\|_1$

 $||x||_1 = ||x - z_S + z_S||_1 \le ||x - z_S||_1 + ||z_S||_1 = ||v_S||_1 + ||z_S||_1 < ||v_{\bar{S}}||_1 + ||z_S||_1$ (because of the NSP)





COHERENCE: DEFINITION

- ▶ Let $A \in \mathbb{R}^{MN}$ be a matrix with normalized columns ($||a_i||_2 = 1 \forall i$).
- The coherence $\mu = \mu(A)$ of the matrix A is given by

• Remark: $\mu \leq 1$

 $\mu = \max_{i \neq j} |\langle a_i, a_j \rangle|$



COHERENCE: PROPERTIES

Let $A \in \mathbb{R}^{MN}$ be a matrix with normalized columns. Then

The equality holds iff A is an equi-angular tight frame (ie $\langle a_i, a_j \rangle = Cte \ (\forall i \neq j)$)

$$\mu \ge \sqrt{\frac{N-M}{M(N-1)}}$$



COHERENCE AND SPARSE RECOVERY

recovered from y = Ax by the basis pursuit and the matching pursuit algorithms if

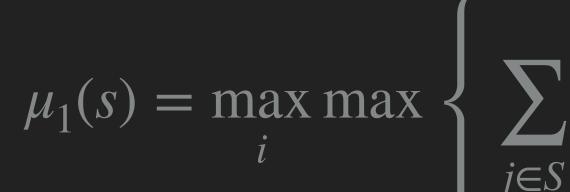
Let $A \in \mathbb{R}^{MN}$ be a matrix with normalized columns. Then, every s-sparse vectors x can be

$$\mu < \frac{1}{2s - 1}$$



BABEL FUNCTION: DEFINITION

- Let $A \in \mathbb{R}^{MN}$ be a matrix with normalized columns ($||a_i||_2 = 1 \forall i$).
- The babel-1 function, or ℓ_1 -coherence $\mu_1(s)$ of the matrix A is given by



Remark: $\mu \leq \mu_1(s) \leq s\mu$

$$|\langle a_i, a_j \rangle|, \operatorname{card}(S) = s, i \notin S$$

BABEL-1 FUNCTION AND SPARSE RECOVERY

Let $A \in \mathbb{R}^{MN}$ be a matrix with normalized columns. Then, every s-sparse vectors x can be recovered from y = Ax by the basis pursuit and the matching pursuit algorithms if

 $\mu_1(s) + \mu_1(s-1) < 1$



PROOF FOR BP

Let $v \in \text{Ker}(A) \setminus \{0\}$. Then $Av = 0 \Leftrightarrow \sum a_i v_i = 0$. Let a set S such that card(S) = s and let $k \in S$. Then $\langle a_k, Av \rangle = \sum v_i \langle a_k, a_i \rangle = 0$, then $v_k = v_k \langle a_k, a_k \rangle =$ Consequently, we have $|v_k| \leq \sum |v_i| |\langle a_i, a_k \rangle|$

$$k \qquad i\in\bar{S} \qquad k$$

Which leads to

$$\|v_S\|_1 < \frac{1 - \mu_1(s - 1)}{\mu_1(s)} \|v_S\|_1 < \|v_{\bar{S}}\|_1, \text{ i.e. } A \text{ satisfield}$$

$$-\sum_{\substack{i \neq k \\ i \in S, i \neq k}} v_i \langle a_i, a_k \rangle = -\sum_{\substack{i \in \bar{S} \\ i \in \bar{S}}} v_i \langle a_i, a_k \rangle - \sum_{\substack{i \in S, i \neq k}} v_i \langle a_i, a_k \rangle$$

Finally $||v||_1 = \sum_k |v_k| < \sum_{i \in \bar{S}} |v_i| \sum_k |\langle a_i, a_k \rangle| + \sum_k \sum_{i \in S, i \neq k} |v_i| \sum_k |\langle a_i, a_k \rangle| < ||v_{\bar{S}}||_1 \mu_1(s) + ||v_{\bar$

es the NSP





BABEL-1 FUNCTION AND SPARSE RECOVERY

Let $A \in \mathbb{R}^{MN}$ be a matrix with normalized columns. Then, every s-sparse vectors x can be recovered from y = Ax by one step of Hard Thresholding if

 $\mu_1(s) + \mu_1(s - s)$

Where S = supp(x)

$$1) < \frac{\min_{i \in S} |x_i|}{\max_{i \in S} |x_i|}$$



PROOF

• We have to show that $\forall k \in S, \forall j \in \overline{S} |\langle a_k, y \rangle| > |\langle a_j, y \rangle|$, ie $|\langle Ax, a_k \rangle| > |\langle Ax, a_j \rangle|$ $|\langle Ax, a_j \rangle| = |\sum_{i \in S} x_i \langle a_i, a_j \rangle| \le \sum_{i \in S} |x_i| |\langle a_i, a_j \rangle| \le \mu_1(s) \max_{i \in S} |x_i|$ $|\langle Ax, a_k \rangle| = |\sum_{i \in S} x_i \langle a_i, a_k \rangle| \ge |x_j| - \sum_{i \in S, i \neq j} |x_i| |\langle a_i, a_k \rangle| \ge \min_{i \in S} |x_i| - \mu_1(s-1) \max_{i \in S} |x_i|$

Then $|\langle Ax, a_k \rangle| - |\langle Ax, a_j \rangle| \ge \min_{i \in S} |x_i| - (\mu_1(s) + \mu_1(s-1)) \max_{i \in S} |x_i| > 0$



BABEL-1 FUNCTION AND SPARSE RECOVERY

Let $A \in \mathbb{R}^{MN}$ be a matrix with normalized columns. Then, every s-sparse vectors x can be recovered from y = Ax by at most s iterations of the iterated Hard Thresholding if

 $2\mu_1(s) + \mu_1(s-1) < 1$

Where S = supp(x)

► IHT:

Where $\mathcal{H}_{s}(x)$ keeps the s largest magnitude value of x

 $x^{(t+1)} = \mathscr{H}_{s} \left(x^{(t)} + A^{*}(y - Ax^{(t)}) \right)$





RESTRICTED ISOMETRY CONSTANT (RIC)

The sth restricted isometry constant δ_s of a matrix $A \in \mathbb{R}^{MN}$ is the smallest $\delta \geq 0$ such that

For all *s*-sparse vector $x \in \mathbb{R}^N$

Equivalently

$(1 - \delta) \|x\|^2 \le \|Ax\|^2 \le (1 + \delta) \|x\|^2$

$\delta_s = \max_{\substack{S, \mathsf{card}(S) \le s}} \|A_s^* A_s - Id\|$



RIC

- The RIC δ_s are increasing
- If A has normalized columns, a coherence μ and a Babel-1 function μ_1 , then

 $\delta_1 = 0, \delta_2 = \mu, \delta_s \leq \mu_1(s-1)$



RESTRICTED ISOMETRY PROPERTY (RIP) AND BP

• Let A such that $\delta_{2s} \leq \frac{1}{3}$

▶ Then every *s*−sparse vector *x* is the unique solution of

 \mathcal{Z}

 $\min \|z\|_1 \quad \text{s.t.} \quad Ax = Az$



RESTRICTED ISOMETRY PROPERTY (RIP) AND IHT

• Let A such that
$$\delta_{3s} \leq \frac{1}{2}$$

- Then, for every s—sparse vector x such that converges to x
- ► IHT:

 $x^{(t+1)} = \mathscr{H}_s\left(.\right)$

Where $\mathscr{H}_{s}(x)$ keeps the *s* largest magnitude value of x

Then, for every s-sparse vector x such that y = Ax, the IHT initialized by a s- sparse vector

$$x^{(t)} + A^*(y - Ax^{(t)})$$



COHERENCE OR RIP ?

- Coherence: easy to check, but very restrictive (few matrix satisfies coherence properties)
- RIP: difficult to check. Satisfied with some random matrices
- Application: compressive sensing

